Physics Letters B 664 (2008) 107–111

Contents lists available at ScienceDirect

Physics Letters B

www.elsevier.com/locate/physletb

Enlarged NH symmetries: Particle dynamics and gauge symmetries

Joaquim Gomis [a], Jerzy Lukierski [b,*]

[a] Departament d'Estructura i Constituents de la Matèria, Institut de Ciencies del Cosmos, Universitat de Barcelona, Diagonal 647, 08028 Barcelona, Spain
[b] Institute of Theoretical Physics, Wrocław University, 50-204 Wrocław, Poland

ARTICLE INFO

Article history:
Received 5 April 2008
Accepted 2 May 2008
Available online 8 May 2008
Editor: M. Cvetič

ABSTRACT

We show how the Newton–Hooke (NH) symmetries, representing a nonrelativistic version of de-Sitter symmetries, can be enlarged by a pair of translation vectors describing in Galilean limit the class of accelerations linear in time. We study the Cartan–Maurer one-forms corresponding to such enlarged NH symmetry group and by using cohomological methods we determine the general 2-parameter (in $D=2+1$ 4-parameter) central extension of the corresponding Lie algebra. We derive by using non-linear realizations method the most general group—invariant particle dynamics depending on two (in $D=2+1$ on four) central charges occurring as the Lagrangean parameters. Due to the presence of gauge invariances we show that for the enlarged NH symmetries quasicovariant dynamics reduces to the one following from standard NH symmetries, with one central charge in arbitrary dimension D and with second exotic central charge in $D=2+1$.

1. Introduction

In the classification of all kinematical groups in $D=3+1$ one finds the two nonrelativistic counterparts of dS and AdS symmetries—two Newton–Hooke (NH) cosmological groups generated by two NH nonrelativistic algebras [1–4]. In comparison with the Galilei algebra, described by the generators I_{ij} (nonrelativistic rotation), P_i (space translations), K_i (nonrelativistic boots) and H (time translations) in NH algebra one relation is deformed for the finite value R of nonrelativistic $(A)dS$ radius

$$[H,P_i]=\pm\frac{K_i}{R^2}\quad\begin{pmatrix}+: & dS\text{ case}\\ -: & AdS\text{ case}\end{pmatrix}. \tag{1.1}$$

In arbitrary dimension D one can introduce one central charge described by the mass parameter m

$$[P_i,K_j]=m\delta_{ij}. \tag{1.2}$$

If $D=2+1$ one can add second "exotic" central charge θ [3,5–9]

$$[K_i,K_j]=\theta\epsilon_{ij},\qquad [P_i,P_j]=-\frac{\theta}{R^2}\epsilon_{ij}. \tag{1.3}$$

Recently the NH symmetries were enlarged by adding the constant accelerations, described by the generator F_i [10]. We denote corresponding algebra by $\widehat{\text{NH}}$ and list below the relations satisfied by new generators F_i, in arbitrary dimension D with central extensions included:

$$[I_{ij},F_k]=\delta_{ik}F_j-\delta_{jk}F_i, \tag{1.4a}$$

$$[H,F_i]=2K_i, \tag{1.4b}$$

$$[P_i,F_j]=[F_i,F_j]=0, \tag{1.4c}$$

$$[K_i,F_j]=\mp 2mR^2\delta_{ij}\quad(-:\quad dS,+:\quad AdS). \tag{1.4d}$$

We see that the relation (1.4d) does not provide a finite limit $R\to\infty$; in order to obtain such a Galilean limit we should introduce R-dependent Newton–Hooke mass parameter

$$m\longrightarrow m(R)=\mp\frac{c}{R^2}. \tag{1.5a}$$

In such a way the relation (1.2) takes the form

$$[P_i,K_j]=\mp\frac{c}{R^2}\delta_{ij} \tag{1.5b}$$

and in the Galilean limit we obtain

$$[P_i,K_j]=0,\qquad [K_i,F_j]=2c, \tag{1.6}$$

in accordance with the formulae introduced in [11]. Further if $D=2+1$, one can introduce two "exotic" central charges θ, θ', satisfying besides the relations (1.3) the following relations [10]

$$[P_i,F_j]=2\theta\epsilon_{ij}, \tag{1.7a}$$

$$[F_i,F_j]=\theta'\epsilon_{ij}. \tag{1.7b}$$

One of the aims of this Letter is to enlarge the NH algebra by adding another vectorial Abelian generator R_i, and introduce a new extension of NH algebra, which we call doubly enlarged $\widehat{\widehat{\text{NH}}}$ algebra. The new algebra describe the symmetries of a class of nonrelativistic reference frames which are characterized also by a relative acceleration linear in time in the flat limit $R\to\infty$.

* Corresponding author.
E-mail addresses: gomis@ecm.ub.es (J. Gomis), lukier@ift.uni.wroc.pl (J. Lukierski).

0370-2693/$ – see front matter © 2008 Published by Elsevier B.V.
doi:10.1016/j.physletb.2008.05.005

We consider a group element of this algebra and we construct the Cartan–Maurer one-forms[1] from which we can calculate the nontrivial Eilenberg–Chevalley cohomology, providing the nontrivial closed invariant two forms which determine the central extensions.[2] In any dimension D the $\widehat{\widehat{\mathrm{NH}}}$ Lie algebra has two central charges, and the number of central charges in $D=2+1$, similarly like for $\widehat{\mathrm{NH}}$ Lie algebra [10], is equal to four.[3]

We further study the physical consequences of the additional generators F_i, R_i. We construct the action of a particle with $\widehat{\widehat{\mathrm{NH}}}$ symmetries, by the method of nonlinear realizations [14]. The action can be written as sum of WZ terms multiplied by central charge parameters. For a generic dimension D the Lagrangean depends on two parameters, but exceptionally if $D=2+1$ it depends on four parameters. The parameters appearing in the Lagrangean are (besides the dS radius R) in one to one correspondence with the central extensions of the $\widehat{\widehat{\mathrm{NH}}}$ algebra.

The consistency of the field equations imply the existence of one (two in $D=2+1$) relations among the two (four) dimensionfull Lagrangean parameters. These constraints permit the existence of two local gauge invariances of the Lagrangean. If a suitable gauge fixing is chosen the gauge-fixed action can be identified with the one considered in [3,9]. We also consider the restriction to the first order $\widehat{\mathrm{NH}}$ quasi-Lagrangean by imposing the vanishing of the Golstone bosons associated to generators $\vec{R}$. If we impose some covariant relations representing inverse Higgs mechanism [15] we recover the $\widehat{\mathrm{NH}}$ quasi-invariant higher Lagrangean of Ref. [10]

The plan of the Letter is as follows: in Section 2 we will introduce the double enlarged $\widehat{\widehat{\mathrm{NH}}}$ algebra and its central extension; the case $D=2+1$ is considered separately. In Section 3 we construct the most general $\widehat{\widehat{\mathrm{NH}}}$ invariant first order Lagrangean and its gauge invariances. We also show that the higher order actions considered in [10,11] can be derived from our linear action. Finally in Section 4 we provide some conclusions.

2. Doubly enlarged Newton–Hooke symmetries

The nonrelativistic de-Sitter and anti-de-Sitter symmetries (see (1.1)) describe respectively the nonrelativistic expanding and nonrelativistic oscillating universes. The cosmological constant $\kappa=\frac{1}{R}$ describes the time scale determining the rate of expansion or the period of oscillations of the universe. When $\kappa\to 0$ we obtain the Galilean group and the standard flat nonrelativistic space–time.

The Newton–Hooke algebra describing nonrelativistic (anti-)de-Sitter symmetries [9,17] was extended in [11] to acceleration—extended Newton–Hooke algebra (see (1.1)–(1.6)). The change of nonrelativistic space–time $(\vec{x}=(x_1,\dots,x_d),t)$ under the transformations of the acceleration—extended NH group are the following [11]

$$\delta x_i = a_i\cosh\frac{t}{R} + v_i R\sinh\frac{t}{R} + 2b_iR^2\left(\cosh\frac{t}{R}-1\right) + \alpha_{ij}x_j,$$
$$\delta t = t + a_0 \tag{2.1}$$

with the following assignments of the parameters

a_i spatial translations (generators P_i), dimension $[a_i]=L^1$,
v_i boosts (generators K_i), dimensions $[v_i]=L^0$,
b_i accelerations (generators F_i), dimensions $[b_i]=L^{-1}$,
α_{ij} $O(d)$ space rotations (generators I_{ij}), dimensions $[\alpha_{ij}]=L^0$,
a_0 time translation (generator H), dimension $[a_0]=L^1$.

Let us observe that in the flat limit $R\to\infty$ we obtain from (2.1) the following formulae for acceleration—enlarged Galilean transformations

$$\delta x_i = a_i + v_it + b_it^2 + \alpha_{ij}x_j,$$
$$\delta t = t + a_0. \tag{2.2}$$

In this Letter we propose to extend the formula (2.2) by adding subsequent term

$$\delta x_i = c_it^3, \tag{2.3}$$

or for $R<\infty$,

$$\delta x_i = 6c_iR^3\left(\sinh\frac{t}{R}-\frac{t}{R}\right). \tag{2.4}$$

The transformations (2.1) and (2.4) lead to the following differential realization of the doubly enlarged $\widehat{\widehat{\mathrm{NH}}}$ algebra on nonrelativistic space–time

$$P_i=\cosh\frac{t}{R}\frac{\partial}{\partial x_i},\qquad K_i=R\sinh\frac{t}{R}\frac{\partial}{\partial x_i},$$
$$F_i=2R^2\left(\cosh\frac{t}{R}-1\right)\frac{\partial}{\partial x_i},$$
$$R_i=6R^3\left(\sinh\frac{t}{R}-\frac{t}{R}\right)\frac{\partial}{\partial x_i},$$
$$H=\frac{\partial}{\partial t},\qquad J_{ij}=x_i\frac{\partial}{\partial x_j}-x_j\frac{\partial}{\partial x_i}. \tag{2.5}$$

The algebra of these vector fields gives the unextended version of the algebra $\widehat{\widehat{\mathrm{NH}}}$[4]

$$[H,P_i]=\frac{K_i}{R^2},\qquad [H,K_i]=P_i,\qquad [H,F_i]=2K_i,$$
$$[H,R_i]=3F_i,\qquad [I_A,I_B]=0,\quad I_A=(P_i,K_i,F_i,R_i),$$
$$[I_{ij},I_k]=\delta_{ik}I_k-\delta_{jk}I_i,\quad I_i=P_i \text{ or } K_i \text{ or } F_i \text{ or } R_i,$$
$$[I_{ij},I_{kl}]=\delta_{ik}I_{jl}-\delta_{il}I_{jk}+\delta_{jl}I_{ik}-\delta_{jk}I_{il}. \tag{2.6}$$

Let us write this Lie algebra in terms of the dual formulation. We introduce the Lie-valued left-invariant Maurer–Cartan (MC) 1-form

$$\Omega_1=g^{-1}dg=L^HH+L_i^PP_i+L_i^KK_i+L_i^FF_i+L_l^RR_l+L_{ij}^II_{ij} \tag{2.7}$$

where g is a general element of the group and L^H, L_i^P, L_i^K, L_i^F, L_l^R, L_{ij}^I are 1-forms, whose explicit form depends on the parametrization of the group element. The MC one-form verifies the MC equation $d\Omega+\Omega\wedge\Omega=0$, in terms of the 1-forms L's we have

$$dL^H=0,$$
$$dL_i^P=L_j^P\wedge L_{ij}^I-L^H\wedge L_l^K,$$
$$dL_i^K=L_i^K\wedge L_{ij}^I-\frac{1}{R^2}L^H\wedge L_i^P-2L^H\wedge L_i^F,$$
$$dL_i^F=L_i^F\wedge L_{ij}^I-3L^H\wedge L_i^R,$$
$$dL_i^R=L_i^R\wedge L_{ij}^I,\qquad dL_{ij}^I=L_{ik}^I\wedge L_{kj}^I. \tag{2.8}$$

If $D=2+1$ the rotations I_{ij} reduce to Abelian rotation $I_{ij}=\epsilon_{ij}I$, and the relations (2.8) take the following form:

[1] Some of the calculations with forms have being done using the Mathematica code for differential forms EDC [12].

[2] See for example [13].

[3] We disregard here in $D=2+1$ the "trivial" central extension due to the deformation of the commutator $[H,I]$, where $I_{ij}=\epsilon_{ij}I$.

[4] We consider the hyperbolic dS case (see (1.1)). The trigonometric case is obtained by changing, $R\to iR$ in all the equations.

$$
\begin{aligned}
&dL^H = 0,\\
&dL_i^P = \epsilon_{ij} L_j^P \wedge L^I - L^H \wedge L_l^K,\\
&dL_i^K = \epsilon_{ij} L_i^K \wedge L^I - \frac{1}{R^2} L^H \wedge L_i^P - 2L^H \wedge L_i^F,\\
&dL_i^F = \epsilon_{ij} L_i^F \wedge L^I - 3L^H \wedge L_i^R,\\
&dL_i^R = \epsilon_{ij} L_i^R \wedge L^I, \qquad dL^I = 0
\end{aligned}
\tag{2.9}
$$

where $L_{ij}^I = \epsilon_{ij} L^I$.

Now we will construct the extended algebra $\widehat{\widehat{\mathrm{NH}}}$. In order to describe the central charges in arbitrary dimension one should look for the invariant closed 2-forms Ω_2, with closure property $d\Omega_2 = 0$ following from the relations (2.8), (2.9), and write the 2-form, $\Omega_2 = d\Omega_1$ as bilinear exterior product of the MC 1-forms (see e.g. [13]). One obtains in arbitrary dimension D the following two nontrivial cohomology classes of rank 2:

$$
\Omega_2^{(c)} = L_F^i \wedge L_K^i - \frac{1}{2R^2} L_K^i \wedge L_P^i + 3L_P^i \wedge L_R^i,
\tag{2.10a}
$$

$$
\Omega_2^{(M)} = L_F^i \wedge L_R^i.
\tag{2.10b}
$$

In $D = 2+1$ one obtain besides the cohomology classes (2.10a)–(2.10b) still two "exotic" cohomology classes, namely

$$
\begin{aligned}
\tilde{\Omega}_2^{(\theta)} = {}& -\frac{1}{2R^2}\epsilon_{ij} L_i^P \wedge L_j^P + \frac{1}{2}\epsilon_{ij} L_i^K \wedge L_j^K - 2\epsilon_{ij} L_i^F \wedge L_j^P\\
&- 2R^2\big(\epsilon_{ij} L_i^F \wedge L_j^F - 3\epsilon_{ij} L_i^K \wedge L_j^R\big),
\end{aligned}
\tag{2.11a}
$$

$$
\tilde{\Omega}_2^{(\theta'')} = \epsilon_{ij} L_i^R \wedge L_j^R.
\tag{2.11b}
$$

In order to obtain the Liouville one-forms $\Theta_1^{(c)}, \Theta_1^{(M)}, \tilde{\Theta}_1, \tilde{\Theta}_1''$

$$
\begin{aligned}
&\Omega_2^{(c)} = d\Theta_1^{(c)}, \qquad \Omega_2^{(M)} = d\Theta_1^{(M)},\\
&\tilde{\Omega}_2^{(\Theta)} = d\tilde{\Theta}_1, \qquad \tilde{\Omega}_2^{(\Theta'')} = d\tilde{\Theta}_1''
\end{aligned}
\tag{2.11c}
$$

one should provide a parametrization of the group element.

From the forms (2.10a)–(2.10b) we can construct the centrally extended Lie algebra in arbitrary dimension D. The algebra is given by $\widehat{\mathrm{NH}}$ relations (2.6) with the commuting algebra of generators (P_i, K_i, F_i, R_i) replaced by the non-vanishing commutation relations:

$$
[P_i, R_j] = -3c\delta_{ij},
\tag{2.12a}
$$

$$
[K_i, P_j] = \frac{1}{2R^2} c\delta_{ij},
\tag{2.12b}
$$

$$
[F_i, K_j] = -c\delta_{ij},
\tag{2.12c}
$$

$$
[F_i, R_j] = -M\delta_{ij}
\tag{2.12d}
$$

where c and M are two central charges.

If $D = 2+1$ from (2.11a)–(2.11b) we obtain two additional central charges θ, θ'':

$$
[F_i, F_j] = 4R^2\theta\epsilon_{ij},
\tag{2.13a}
$$

$$
[P_i, P_j] = \frac{1}{R^2}\theta\epsilon_{ij},
\tag{2.13b}
$$

$$
[F_i, P_j] = 2\theta\epsilon_{ij},
\tag{2.13c}
$$

$$
[K_i, K_j] = -\theta\epsilon_{ij},
\tag{2.13d}
$$

$$
[K_i, R_j] = -6R^2\theta\epsilon_{ij},
\tag{2.13e}
$$

$$
[R_i, R_j] = -\theta''\epsilon_{ij}.
\tag{2.13f}
$$

We see therefore that our new enlarged $\widehat{\widehat{\mathrm{NH}}}$ algebra in $D = 2+1$ depends on two standard (m and M) and two exotic (θ and θ'') central charges.

3. Particle Lagrangean from nonlinear realizations

In this section we will construct the most general particle Lagrangean (quasi-)invariant under the group $\widehat{\widehat{\mathrm{NH}}}$ and its gauge invariances. We will use the method of nonlinear realizations [14]. As we shall see the most general $\widehat{\widehat{\mathrm{NH}}}$ first order invariant action is endowed with gauge symmetries. We also discuss the corresponding $\widehat{\mathrm{NH}}$ first order quasi-invariant Lagrangean as well as the connection with the higher order Lagrangean proposed in [10].

3.1. General reparametrization invariant action

Let us consider the coset of the unextended algebra $\frac{\widehat{\mathrm{NH}}}{SO(d)}$. We locally parametrize the coset as follows

$$
g = e^{tH} e^{\vec{x}\vec{P}} e^{\vec{v}\vec{K}} e^{\vec{w}\vec{F}} e^{\vec{s}\vec{R}}.
\tag{3.1}
$$

The (Goldstone) coordinates of the coset depend on the parameter τ that parametrizes the world line of the particle (see for example [16]). The corresponding Maurer–Cartan one-form

$$
\Omega_1 = L^H H + L_i^P P_i + L_i^K K_i + L_i^F F_i + L_l^R R_l + L_{ij}^I I_{ij}.
\tag{3.2}
$$

In arbitrary dimension D the 1-forms L are

$$
\begin{aligned}
&L^H = dt,\\
&L_i^P = v_i\, dt + dx_i,\\
&L_i^K = dv_i + 2w_i\, dt + \frac{1}{R^2} x_i\, dt,\\
&L_i^F = dw_i + 3s_i\, dt,\\
&L_i^R = ds_i.
\end{aligned}
\tag{3.3}
$$

Substituting the one-forms in the formulae (2.10a), (2.10b) and (2.11a), (2.11b) one obtains the following formulae for the corresponding Liouville one-forms, satisfying the relations (2.11c)

$$
\begin{aligned}
\Theta_1^{(c)} = {}& \vec{w}^2\, dt - \frac{1}{4R^2}\vec{v}^2\, dt + \frac{\vec{w}\cdot\vec{x}}{R^2} dt\\
&+ \frac{1}{4}\frac{1}{(R^2)^2}\vec{x}^2\, dt - \frac{1}{2R^2}\vec{v}\cdot d\vec{x} + \vec{w}\, d\vec{v}\\
&- 3\vec{s}\, d\vec{x} - 3\vec{s}\cdot\vec{v}\, dt,
\end{aligned}
\tag{3.4a}
$$

$$
\Theta_1^{(M)} = -\vec{s}\, d\vec{w} - \frac{3}{2}\vec{s}^2\, dt
\tag{3.4b}
$$

and

$$
\begin{aligned}
\tilde{\Theta}_1 = {}& \epsilon_{ij}\left(\frac{1}{2} v_i\, dv_j - 2w_i\, dx_j\, dt + 2v_i w_j\, dt - 6x_i s_j\, dt\right)\\
&+ \frac{\epsilon_{ij}}{R^2}\left(v_i x_j\, dt - \frac{1}{2} x_i\, dx_j\right)\\
&- 2\epsilon_{ij} R^2 (w_i\, dw_j - 3s_i\, dv_j - 6s_i w_j\, dt),
\end{aligned}
\tag{3.5a}
$$

$$
\tilde{\Theta}_1'' = \frac{1}{2}\epsilon_{ij} s_i\, ds_j.
\tag{3.5b}
$$

In arbitrary dimension D the action is given by

$$
S = \int \big(c\Theta_1^{(C)} + M\Theta_1^{(M)}\big)^*
\tag{3.6}
$$

and in the $D = 2+1$ we have

$$
\tilde{S} = S + \int \big(\theta\tilde{\Theta}_1 + \theta''\tilde{\Theta}_1''\big)^*
\tag{3.7}
$$

where * means pullback on the worldline of the particle parametrized by τ. One can fix the diffeomorphism invariance by choosing $t = \tau$. In D dimensions the Lagrangean deduced from (3.6) has the following explicit form

$$L_1 = c\left[\left(\vec{w}^2 - \frac{1}{4R^2}\vec{v}^2 + \frac{\vec{w}\cdot\vec{x}}{R^2} + \frac{1}{4}\frac{1}{(R^2)^2}\vec{x}^2 - 3\vec{s}\cdot\vec{v}\right) - \left(3\vec{s} + \frac{1}{2R^2}\vec{v}\right)\dot{\vec{x}} + \vec{w}\cdot\dot{\vec{v}}\right] - M\left(\vec{s}\dot{\vec{w}} + \frac{3}{2}\vec{s}^2\right). \tag{3.8}$$

The field equations following from (3.8) are

$$\frac{1}{2R^2}\left(\frac{1}{R^2}x_i + \dot{v}_i\right) + \frac{w_i}{R^2} + 3\dot{s}_i = 0, \tag{3.9a}$$

$$\frac{1}{2R^2}(\dot{x}_i + v_i) + (3s_i + \dot{w}_i) = 0, \tag{3.9b}$$

$$c\left(\frac{1}{R^2}x_i + \dot{v}_i\right) + 2cw_i + M\dot{s}_i = 0, \tag{3.9c}$$

$$M(\dot{w}_i + 3s_i) + 3c(v_i + \dot{x}_i) = 0. \tag{3.9d}$$

3.2. Gauge invariances and reduction to oscillator dynamics

From relations (3.9a) and (3.9c) follows as the consistency condition (if $\dot{s}_i \neq 0$)

$$M = 6cR^2. \tag{3.10}$$

If (3.10) is valid the pair of relations (3.9c), (3.9d) are the same as the relations (3.9a), (3.9d). Eqs. (3.9a), (3.9b) describe the following relations between the one-forms (3.3) ($L^x = \dot{L}^x\,dt$, where $\dot{L}^x = \frac{dL^x}{dt}$)

$$\frac{1}{2R^2}\dot{L}_i^P + \dot{L}_i^F = 0, \tag{3.11a}$$

$$\frac{1}{2R^2}\dot{L}_i^K + 3\dot{L}^R = 0. \tag{3.11b}$$

We see that in field equations (3.11a)–(3.11b) out of four functions $x_i(t)$, $v_i(t)$, $w_i(t)$, $s_i(t)$ two can be chosen arbitrary. Such a property reflects the presence (if condition (3.10) is satisfied) of two-parameter local gauge invariance. Indeed, the action (3.8) depends only on the following two linear combinations of the field variables.

$$u_i = w_i + \frac{x_i}{2R^2}, \tag{3.12a}$$

$$y_i = s_i + \frac{1}{6R^2}v_i. \tag{3.12b}$$

Substituting (3.12a)–(3.12b) into (3.8) one obtains

$$L_1 = c\left(\vec{u}^2 - 9R^2\vec{y}^2 - 6R^2 y_i\cdot\dot{u}_i\right). \tag{3.13}$$

The dependence of L on linear combinations (3.12a), (3.12b) means that the action (3.8) is invariant under the following two local gauge transformations, leaving the variables u_i and y_i invariant

$$\delta x_i = \epsilon_i, \qquad \delta w_i = -\frac{1}{2}\frac{\epsilon_i}{R^2},$$
$$\delta s_i = \eta_i, \qquad \delta v_i = -\frac{1}{6}\frac{\eta_i}{R^2}. \tag{3.14}$$

By fixing the gauge invariances (3.14)

$$\epsilon_i = 2R^2 w_i \quad\Rightarrow\quad u_i = \frac{1}{2R^2}x_i, \quad w_i = 0, \tag{3.15a}$$

$$\eta_i = 6R^2 s_i \quad\Rightarrow\quad y_i = \frac{1}{6R^2}v_i, \quad s_i = 0 \tag{3.15b}$$

we obtain the following gauge-fixed form of the action (3.8)

$$L_1^{\text{fix}} = -\frac{c}{2R^2}\left(\vec{v}\dot{\vec{x}} + \frac{1}{2}\vec{v}^2 - \frac{1}{2}R^2\vec{x}^2\right). \tag{3.16}$$

The field equations derived from (3.16) are

$$\vec{v} + \dot{\vec{x}} = 0, \qquad \dot{\vec{v}} + \frac{1}{R^2}\vec{x} = 0, \tag{3.17}$$

lead to the hyperbolic oscillator equation

$$\ddot{\vec{x}} - \frac{1}{R^2}\vec{x} = 0 \tag{3.18}$$

with the frequency described by the inverse of dS radius.

In $D = 2+1$ dimensions from (3.5a)–(3.5b) one obtains the following part of the action depending on two exotic charges θ, θ''

$$L_2 = \theta\left[\frac{1}{2}\epsilon_{ij}v_i\dot{v}_j - \frac{1}{2R^2}\epsilon_{ij}x_i\dot{x}_j - 2\epsilon_{ij}w_i\dot{x}_j + 2\epsilon_{ij}v_iw_j - 6\epsilon_{ij}x_is_j + \frac{\epsilon_{ij}}{R^2}v_ix_j - 2\epsilon_{ij}R^2(w_i\dot{w}_j - 3s_i\dot{v}_j - 6s_iw_j)\right] + \frac{\theta''}{2}\epsilon_{ij}s_i\dot{s}_j. \tag{3.19}$$

Introducing new variable u_i (see (3.12a)) one gets

$$L_2 = \theta\left(-2\epsilon_{ij}R^2u_i\dot{u}_j + 2\epsilon_{ij}v_iu_j + 12\epsilon_{ij}R^2s_iu_j + \frac{1}{2}\epsilon_{ij}v_i\dot{v}_j + 6\epsilon_{ij}R^2s_i\dot{v}_j\right) + \frac{\theta''}{2}\epsilon_{ij}s_i\dot{s}_j, \tag{3.20}$$

the field equations are

$$2\theta\epsilon_{ij}\left(2R^2\dot{u}_j + v_j + 6R^2s_j\right) = 0, \tag{3.21a}$$

$$\epsilon_{ij}\left(\dot{v}_j + 2u_j + 6R^2\dot{s}_j\right) = 0, \tag{3.21b}$$

$$6\theta R^2\epsilon_{ij}(\dot{v}_j + 2u_j) + \theta''\epsilon_{ij}\dot{s}_j = 0, \tag{3.21c}$$

by consistency (if $\dot{s}_i \neq 0$) we have

$$\theta'' = 36R^4\theta. \tag{3.22}$$

In such a case the action (3.20) depends only on two variables (3.12a)–(3.12b)

$$L_2 = \theta R^2\left(-2\epsilon_{ij}u_i\dot{u}_j + 12\epsilon_{ij}y_iu_j + 18\epsilon_{ij}R^2y_i\dot{y}_j\right). \tag{3.23}$$

The most general action in $D = 2+1$ is described by the sum of the action (3.8) and (3.19). If we write down the field equations it appears that one can derive the following equation

$$A\dot{s}_i + B\epsilon_{ij}\dot{s}_j = 0 \tag{3.24}$$

where

$$A = M - 6cR^2, \qquad B = \theta'' - 36R^4\theta. \tag{3.25}$$

The solvability of Eq. (3.24) leads to

$$\det\begin{pmatrix} A & B \\ -B & A \end{pmatrix} = 0 \quad\Rightarrow\quad A^2 + B^2 = 0$$
$$\Rightarrow\quad A = 0, \quad B = 0 \tag{3.26}$$

i.e. we reproduce the constraints (3.10) and (3.22).

Using the conditions (3.15a)–(3.15b) and (3.10), (3.22) one obtains the following general gauge-fixed action in $D = 2+1$

$$L^{\text{fix}} = L_1^{\text{fix}} + L_2^{\text{fix}} = -\frac{c}{4R^2}\left(2v_i\dot{x}_i + \vec{v}^2 - \frac{1}{R^2}\vec{x}^2\right) + \theta\epsilon_{ij}\frac{1}{2}\left(-\frac{x_i\dot{x}_j}{R^2} - 2\frac{x_iv_j}{R^2} + v_i\dot{v}_j\right) \tag{3.27}$$

which can be identified with the one presented in [9] (see formulas (3.5a), (3.5b)) if we pass from dS to AdS case ($R \to iR$), parametrize velocity with opposite sign ($v_i \to -v_i$) and put $m = -\frac{c}{2R^2}$, $\theta = \frac{1}{2}\kappa$. One can state therefore that the physical content of our model is the same as in the case of standard Newton–Hooke symmetries, with the same number of generators as in Galilean case.

3.3. Relation with $\widehat{\mathrm{NH}}$ higher order Lagrangean

Firstly we derive the $\widehat{\mathrm{NH}}$ first order Lagrangean from our results presented above. For that purpose we should put equal to zero the Goldstone fields associated with generators $\vec{R}$ ($\vec{s}=0$). For dimension $D \neq 2+1$ the action density becomes $L_1|_{\vec{s}=0}$, where the Lagrangean L_1 is given by (3.8). Instead for $D=2+1$ there is an extra term $L_2|_{\vec{s}=0}$ with L_2 described by (3.19).

Let us use the variables

$$u_i = w_i + \frac{x_i}{2R^2}, \tag{3.28}$$

$$y_i = \frac{1}{6R^2} v_i. \tag{3.29}$$

We can write the Lagrangeans as

$$L_1|_{\vec{s}=0} = \tilde{c}\left(2\vec{u}^2 - \frac{1}{2R^2}\vec{v}^2 - 2v_i \cdot \dot{u}_i\right), \tag{3.30}$$

$$L_2|_{\vec{s}=0} = \theta'\left(\epsilon_{ij} u_i \dot{u}_j - \frac{1}{2R^2}\epsilon_{ij} v_i u_j - \frac{1}{4R^2}\epsilon_{ij} v_i \dot{y}_j\right) \tag{3.31}$$

where $\tilde{c} = \frac{c}{2}$, $\theta' = -4R^2\theta$.

If we employ the covariant inverse Higgs mechanism [15], by putting $L_i^K = L_i^P = 0$ and using (3.3), (3.28) we obtain

$$v_i = -\dot{x}_i, \tag{3.32}$$

$$u_i = -\frac{1}{2}\dot{v}_i. \tag{3.33}$$

From (3.30) and (3.31) we get the $\widehat{\mathrm{NH}}$ higher order Lagrangean proposed in [10].

4. Conclusions

We demonstrated that the action constructed via Cartan–Maurer one-forms for extended NH groups with additional constant and linear acceleration generators (F_i, R_i) supplements only a gauge sector of the standard Newton–Hooke dynamics. To achieve such equivalence we have to reduce the number of central charges via suitably chosen relations (3.10) and (3.20). We recall that changing the sign of R^2 (see (1.1)) we obtain from the hyperbolic oscillator (see (3.18)) the standard one, with trigonometric solutions (see [9]).

It seems therefore that if we consider more general NH algebras with extra Abelian generators, the particle dynamics will be equivalent to the one describing ordinary NH particle. This is due the appearance of gauge symmetries associated the extra generators. It will be interesting to understand better such a property.

We have also seen that if we assume that Goldstone fields associated with $\vec{R}$ vanish and we impose some covariant relations one recovers the $\widehat{\mathrm{NH}}$ higher order Lagrangeans introduced in Ref. [10].

Finally we would like to observe that the transformations (2.1)–(2.4), for $R \to \infty$, can be extended as follows

$$x_i' = x_i + a_i(t). \tag{4.1}$$

We get in such a way the reparametrization group of Newton's equation of particle in the presence of nonrelativistic gravitational potential $V(x,t)$ (see e.g. [18]).

$$\frac{d^2 x_i}{dt^2} = \frac{\partial V}{\partial x_i} \tag{4.2}$$

which is invariant under (4.1) if $V' = V - \ddot{a}_i x_i$. It seems interesting to consider the link between the extension of system (4.2) for R finite and the enlarged NH symmetries of arbitrary order.

Acknowledgements

We acknowledge Roberto Casalbuoni, Gary Gibbons, Mikhail Plyushchay, Peter Stichel and Andrzej Trautmann for useful discussions. This work has been supported by the European EC-RTN project MRTN-CT-2004-005104, MCYT FPA 2007-66665, CIRIT GC 2005SGR-00564, Spanish Consolider-Ingenio 2010 Programme CPAN (CSD2007-00042) and by Polish MNiSW grant N N202 318534 (J.L.). We would like to thank the TH Division at CERN for its hospitality where this works has started.

References

[1] H. Bacry, J. Levy-Leblond, J. Math. Phys. 9 (1968) 1605.
[2] H. Bacry, J. Nuyts, J. Math. Phys. 27 (1986) 2455.
[3] Y.h. Gao, hep-th/0107067.
[4] G.W. Gibbons, C.E. Patricot, Class. Quantum Grav. 20 (2003) 5225, hep-th/0308200.
[5] J.-M. Lévy-Leblond, Galilei group and Galilean invariance, in: E.M. Loebl (Ed.), Group Theory and Applications, vol. II, Academic Press, New York, 1972, p. 222.
[6] A. Ballesteros, M. Gadella, M. del Olmo, J. Math. Phys. 33 (1992) 3379.
[7] Y. Brihaye, C. Gonera, S. Giller, P. Kosiński, hep-th/9503046.
[8] O. Arratia, M.A. Martin, M.A. Olmo, math-ph/9903013.
[9] P.D. Alvarez, J. Gomis, K. Kamimura, M.S. Plyushchay, Ann. Phys. 322 (2007) 1556, hep-th/0702014.
[10] J. Lukierski, P.C. Stichel, W.J. Zakrzewski, Eur. Phys. J. C (2008), doi:10.1140/epjc/s10052-008-0576-8, arXiv: 0710.3093 [hep-th].
[11] J. Lukierski, P.C. Stichel, W.J. Zakrzewski, Phys. Lett. B 650 (2007) 203, hep-th/0702179.
[12] S. Bonanos, Exterior Differential Calculus, http://www.inp.demokritos.gr/~sbonano/.
[13] J.A. de Azcarraga, J.M. Izquierdo, Lie groups, Lie Algebras, Cohomology and Some Applications in Physics, Cambridge Univ. Press, 1995.
[14] S.R. Coleman, J. Wess, B. Zumino, Phys. Rev. 177 (1969) 2239;
C.G. Callan, S.R. Coleman, J. Wess, B. Zumino, Phys. Rev. 177 (1969) 2247.
[15] E.A. Ivanov, V.I. Ogievetsky, Teor. Mat. Fiz. 25 (1975) 164.
[16] J. Gomis, K. Kamimura, P. West, Class. Quantum Grav. 23 (2006) 7369, hep-th/0607057.
[17] J. Negro, M.A. del Olmo, M. Santander, J. Math. Phys. 38 (1997) 3786;
J. Negro, M.A. del Olmo, M. Santander, J. Math. Phys. 38 (1997) 3810.
[18] A. Trautmann, Foundations and Current Problems of General Relativity, Brandeis Summer Institute, Lectures on General Relativity, Prentice–Hall, 1965.